ℓ_1 ROBUST CONTROL

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Contents

Summary

In this chapter we present a brief description of ℓ_1 robust control. We first motivate the use of the ℓ_1 norm and then show how it is used in formulating a control problem with the objective of minimizing the effect of unknown by bounded disturbances. This leads to formulation of the ℓ_1 optimal control problem, for which we present a solution using the Scaled-Q method. We then address the problem of robust stability and performance in the face of structured unmodeled dynamics. Here we provide simple and non-conservative conditions for robustness that are expressed in terms of the ℓ_1 norms of the nominal system.

1. Introduction

Feedback control is required only to combat uncertainty. Indeed without uncertainty, feedback control has no advantage over open-loop control. The ubiquitous presence of feedback as a control strategy in engineering systems and natural systems such as biological systems is a reflection of the fact that virtually all systems to be controlled have uncertainty in one form or another. This state of affairs is captured succinctly by Albert Einstein's quote "As far as the laws of mathematics refer to reality, they are not certain; as far as they are certain, they do not refer to reality". In engineering systems, it is useful to characterize this uncertainty and to try to quantify its effect on system stability and performance. Basing any control design on a model of the physical system to be controlled may lead to control designs that are unstable, have poor performance, or very sensitive to system variations and are therefore of little practical use. Thus any practical control design must be robust to the inevitable uncertainty.

For our purpose we will distinguish between two broad classes of uncertainty: signal uncertainty, and model uncertainty. Signal uncertainty refers to those exogenous uncertain signals that impact the control system but do not affect its fundamental characteristics or its internal structure. They are to be viewed as external inputs to which the system will respond, but whose presence does not affect the system dynamics. Examples include wind gusts affecting an airplane in flight, or sensor noise that is superimposed on a given measured variable. Model uncertainty, on the other hand, refers to mismatches between the physical system and the corresponding model. Such mismatches could be in the form of unmodeled or poorly modeled dynamics, including parameter mismatches.

Characterizing and quantifying uncertainty is the basis for robust control analysis and design. When modeling signal uncertainty, it is convenient to consider a class of possible signals that affect the control system under consideration. Typically, only a bound on the size of these signals is available, but otherwise the exact signal that will impact the system is unknown. Choosing the appropriate measure for signal size determines the nature of the set of possible disturbances. Likewise, the effect of these input signals at a given system output (e.g. errors) may be assessed using an appropriate signal measure which may or may not be the same as that used for the input. Signals norms are convenient measures of signal size. For instance the ℓ_2 norm of a signal is a reflection of its energy content. One the other hand the ℓ_∞ norm of a signal captures the maximum peak that the signal achieves over time. Thus the class of signals $\mathcal{W} := \{w : \| w \|_\infty \leq 1\}$ represents the set of signals whose magnitude over time remains less than or equal to unity. Bounded magnitude disturbances arise very frequently in applications, and while the disturbance signal affecting the system may not be known, a bound on its magnitude is typically known. As a practical matter, it is therefore convenient to model uncertain signals as a set of possible signals, e.g. $\mathcal{W}$.

2. The ℓ_1 Norm

Once uncertain signals are modeled by sets, one may ask: “what is the worst case effect of such signals on a given output of interest?” Here again, we find that an appropriate measure of the output signal is needed. There are many possible choices, but norms are especially convenient. Moreover in many applications, the maximum amplitude over time of a output signal of interest, e.g. error, is just the correct measure to use. For example a robot operating in an environment full of obstacles must not have the peak of its tracking errors exceed a certain limit. An aircraft control surface must not be allowed to saturate and its maximum deflection must be restricted to remain below a certain angle. An electric device must not have its input voltage exceed its rated value. So it appears that the ℓ_∞ norm is a particularly attractive measure for output size. If we let e denote the output signal of interest, and if we take $\mathcal{W}$ to represent the uncertain disturbances, then a measure for robustness to these uncertain disturbances is given by the quantity: $\sup_{w\in\mathcal{W}} \frac{\|e\|_\infty}{\|w\|_\infty}$. This has the interpretation of worst-case gain from the disturbance input to the error output. Since e is the output of system $\mathbf{T}$ when the input is w, it can be expressed by $e = \mathbf{T}w$. We shall assume that $\mathbf{T}$ is a stable system (i.e. has a bounded map); otherwise there will exist unbounded outputs e corresponding to bounded inputs w, and the worst-case gain defined above will be infinite. How does one calculate this worst case again, given the system description $\mathbf{T}$? The answer to this questions when $\mathbf{T}$ is a linear time-invariant systems (LTI) involves the system's ℓ_1 norm, which we address next.

Assuming $\mathbf{T}$ is LTI and is stable, its impulse response sequence T will be an ℓ_1 signal and is therefore absolutely summable. In this case $0, e$ is related to w through convolution, $e = T * w$. It turns out that worst case disturbance gain is computed in terms of T as follows

$$\sup_{w\in\mathcal{W}} \frac{\|e\|_\infty}{\|w\|_\infty} = \|T\|_1 \qquad (1)$$

We now turn to verifying the relation (1). Certainly we have for any nonzero $w \in \ell_\infty$

$$|(T*w)(k)| = |\sum_{i=0}^{k} T(i)w(k-i)| \le \sup_i |w(i)| \sum_{i=0}^{\infty} |T(i)| = \|T\|_1 \|w\|_\infty \qquad (2)$$

Furthermore, for any N, the sequence w_N defined by $w_N(i) := \mathrm{sgn}(T(N-i))$ for $0 \le i \le N$ and zero otherwise, satisfies $\|w_N\|_\infty = 1$ and

$$(T*w_N)(N) = \sum_{i=0}^{N} T(i)w_N(N-i) = \sum_{i=0}^{N} |T(i)| \qquad (3)$$

Thus $\|T*w_N\|_\infty \ge \sum_{i=0}^{N} |T(i)|$, which together with equation (2), gives (1).

At this point it is instructive to explore the implication of the relation (1) to the system's robustness. Since $\mathbf{T}$ is the mapping between disturbances and error, the smaller $\|T\|_1$ is, the more attenuated the worst case disturbances will be. Hence, to the extent we can modify $\mathbf{T}$ through a design process, the ℓ_1 norm of its impulse response must be made as small as possible.

3. Robustness to Signal Uncertainty: The ℓ_1 Norm Minimization Problem

It is natural to ask how a given input-output map can be influenced through control design. We can consider the simplest example of output disturbance rejection. The setup appears in figure 1 where w

Is an external ℓ_∞ disturbance, $\mathbf{W}$ is a stable LTI weighting function designed to capture the frequency content of the disturbance, and e is the system output. $\mathbf{P}$ and $\mathbf{K}$ are LTI systems representing the plant and controller, respectively. The objective is to reject the effect of the worst case disturbance on the output, as measured by the ℓ_∞ norm. The map between w is e is given by $\mathbf{T} = (\mathbf{I} + \mathbf{PK})^{-1}\mathbf{W}$. From our previous discussion we would like $\mathbf{T} = (\mathbf{I} + \mathbf{PK})^{-1}\mathbf{W}$ to be stable and have an impulse response T with an ℓ_1 norm as small as possible. In any practical design, stability of $\mathbf{T}$ is not enough. We must enforce the stronger constraint that the entire system interconnection be stable. With this in mind, we now have the basic formulation of an ℓ_1 optimization problem:

$\min \|T\|_1$, subject to $\mathbf{T} = (\mathbf{I} + \mathbf{PK})^{-1}\mathbf{W}$ and $\mathbf{K}$ stabilizes the system interconnection (4)

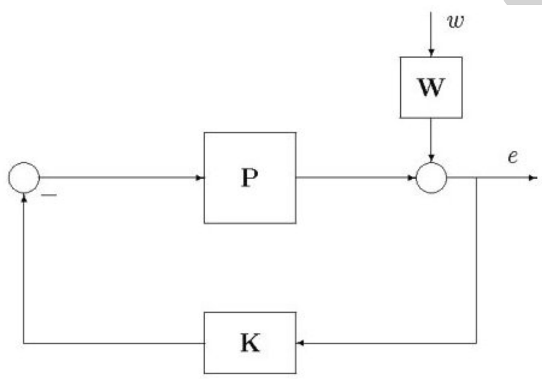

Figure 1: Feedback loop with output disturbance

In order to accommodate a more general setup that the one considered here, we can consider a generalized problem setup as shown in figure 2. $\mathbf{G}$ represent the linear time-invariant (LTI) discrete-time

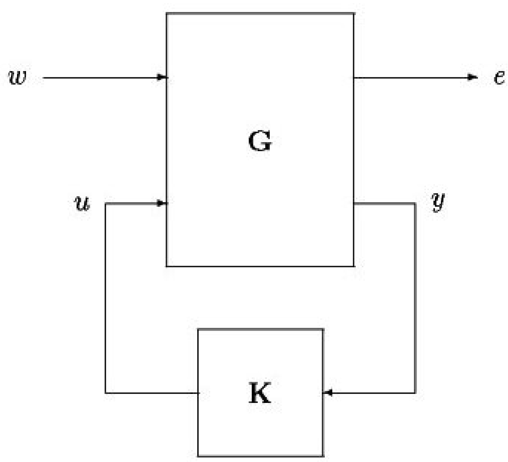

Figure 2: The configuration used in the ℓ_1 problem formulation

discrete-time generalized plant, **K** the LTI discrete-time controller, u the control input, y the measured output, w the external inputs, and e the controlled output. To simplify the exposition, we will assume that w, e, u, and y are all scalars. **G** may be partitioned as follows:

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

The objective is to find an LTI discrete-time controller which minimizes the ℓ_1 norm of the impulse response of the function **T** which maps w to e.

This task can be made more manageable by using a parameterization of all stabilizing controllers in terms of a parameter, $Q \in \ell_1$, and then by expressing T in terms of that parameter.

This parameterization described here is the so-called Youla parameterization of all stabilizing controllers. This elegant parameterization can be summarized in terms of our setup as follows:

Theorem 1 (Youla Parameterization) *Let* **G** *be stabilizable from* u *and detectable from* y *. Find a coprime factorization of* $\hat{G}_{22}$ *as follows:*

$$\hat{G}_{22} = \frac{\hat{N}}{\hat{M}}, \quad \hat{M}\hat{A} - \hat{N}\hat{B} = 1, \quad M, N, A, B \in \ell_1. \tag{5}$$

Then **K** *stabilizes the system if and only if* $\hat{K} = \frac{\hat{B} - \hat{M}\hat{Q}}{\hat{A} - \hat{N}\hat{Q}}$ *for some* $Q \in \ell_1$. *Furthermore, for any such controller* $\hat{T} = \hat{H} - \hat{U}\hat{Q}$, *where* $\hat{H} = \hat{G}_{11} + \hat{G}_{12}\hat{B}\hat{M}\hat{G}_{12}, \hat{U} = \hat{G}_{12}\hat{M}^2\hat{G}_{21}$, *and* $H, U \in \ell_1$.

This result allows us to express our optimization problem as follows:

ℓ_1 **Optimization Problem:** Find $\inf_{Q \in \ell_1} \| H - U * Q \|_1 =: \gamma^{opt}$ (6)

where $H, U \in \ell_1$. Hence the search over stabilizing controllers is replaced by a search over sequences $Q \in \ell_1$. Equally important, T depends in an affine linear manner on the parameter $Q : T = H - U * Q$. The controller impulse response K also depends on the choice of the parameter Q.

It will be assumed that U has finite support, i.e. that $\hat{U}$ is polynomial in z^{-1}. Otherwise if $\hat{U}$ were rational in z^{-1}, its denominator can be absorbed in $\hat{Q}$.

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Biographical Sketch

Mustafa Khammash received his B.S. degree from Texas A&M University in 1986 and his Ph.D. from Rice University in 1990, both in electrical engineering. In September 1990 he joined the Electrical and Computer Engineering Department at Iowa State University where he led the systems and control group until 2002. In May 2002 he joined the Department of Mechanical and Environmental Engineering at the University of California at Santa Barbara where he is currently a Professor and Associate Director of the Center for Control Engineering and Computation. Dr. Khammash's research interests are in the areas of linear and nonlinear robust control, system theory, power systems stability and control, flight control, and Systems Biology. He is the recipient of the National Science Foundation Young Investigator Award, the ISU Foundation Early Achievement in Research and Scholarship Award, the ISU College of Engineering Young Faculty Research Award and the Ralph Budd Best Engineering Ph.D. Thesis Award.